3.5 Curve Sketching

① $f(x) = \dfrac{2x}{x+4}$

Domain: $(-\infty,-4)\cup(-4,\infty)$.

Asymptotes: **Vertical** (where denominator is 0) $\boxed{x=-4}$

$$\lim_{x\to -4^-} \frac{2x}{x+4} = +\infty \qquad \frac{-8}{0_-}$$

$$\lim_{x\to -4^+} \frac{2x}{x+4} = -\infty \qquad \frac{-8}{0_+}$$

Horizontal (limit at ∞) $\boxed{y=2}$

$$\lim_{x\to\infty} \frac{2x}{x+4} = \lim_{x\to\infty} \frac{2}{1+\frac{4}{x}} = 2 \qquad \lim_{x\to-\infty} \frac{2x}{x+4} = 2 \text{ also}$$

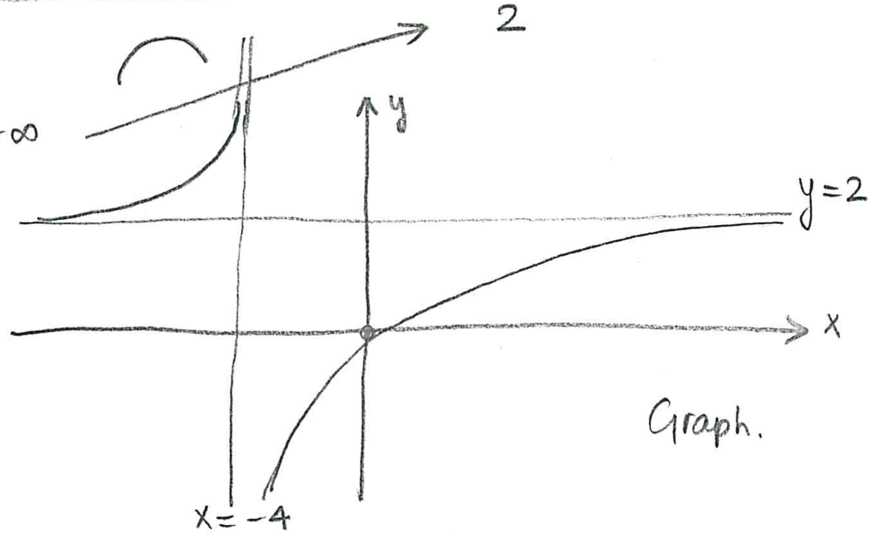
Monotonicity: $f'(x) = \dfrac{2(x+4)-2x\cdot 1}{(x+4)^2} = \dfrac{8}{(x+4)^2} > 0$ for all $x \neq -4$

$\Rightarrow f$ is $\nearrow$ on $(-\infty,-4)\cup(-4,\infty)$.

Concavity: $f'(x) = 8\cdot(x+4)^{-2}$; $f''(x) = -2\cdot 8\,(x+4)^{-3} = \dfrac{-16}{(x+4)^{-3}}$

$(x+4)^{-3}$ is $\begin{cases} \ominus & \text{for } x<-4 \\ \oplus & \text{for } x>-4 \end{cases}$ → Careful though, $x=-4$ is <u>not</u> an inflection pt. b/c $-4 \notin$ Domain !

x	$-\infty$	-4	∞
$f'(x)$	+ + + +		+ + + + + + + + +
$f''(x)$	+ + + +		− − − − − − − −
$f(x)$	2 ↗ ∞ (∪)		$-\infty$ ↗ 2 (∩)



Graph.

② $f(x) = 5x^{4/5} - 4x = 5\sqrt[5]{x^4} - 4x$

Domain: $(-\infty, \infty)$

Asymptotes: no vertical.

horizontal? none

$$\lim_{x\to\infty}\left(5x^{4/5} - 4x\right) = \lim_{x\to\infty} x^{4/5}\left(5 - 4x^{1/5}\right) = \infty(-\infty) = -\infty$$

$$\lim_{x\to-\infty}\left(5x^{4/5} - 4x\right) = \lim_{x\to\infty} \underbrace{x^{4/5}}_{+\infty}\underbrace{\left(5 - 4x^{1/5}\right)}_{+\infty} = \infty(+\infty) = +\infty$$

Monotonicity: $f'(x) = 4x^{-1/5} - 4 = \frac{4}{x^{1/5}} - 4 = \frac{4 - 4x^{1/5}}{x^{1/5}}$

$f'(x)$ dne at $x = 0$ (c.pt.)

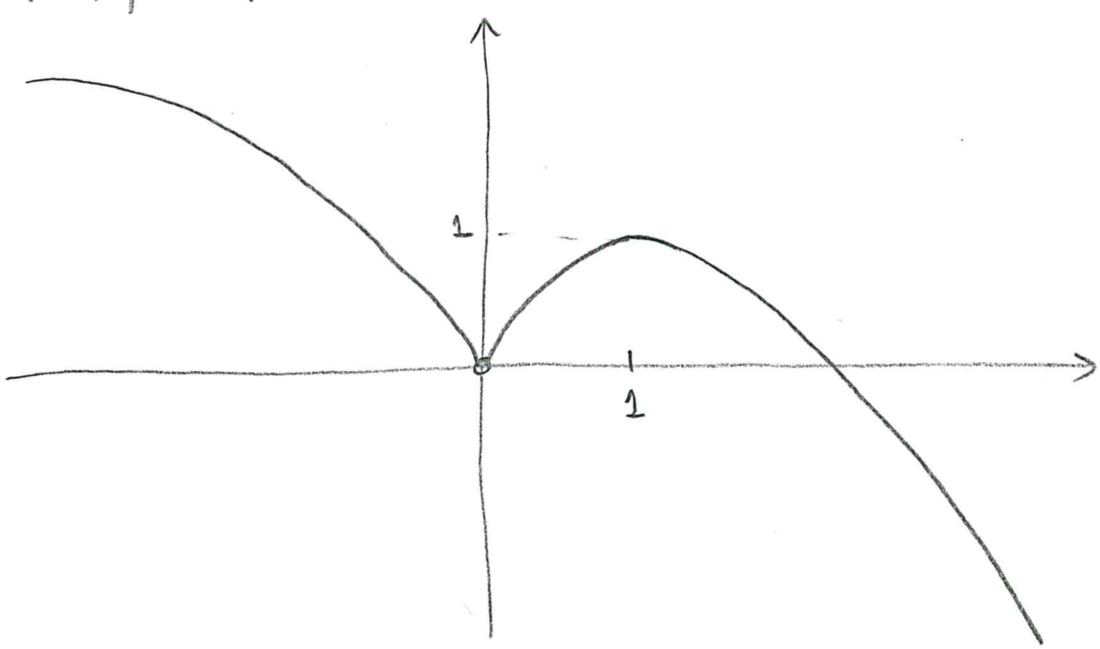
$f'(x) = 0$ if $4 - 4x^{1/5} = 0$, or $1 = x^{1/5}$ or $x = 1$ (c.pt.)

x	$-\infty$		0		1		∞
$f'(x)$		$- - - - -$	$\vert$	$+ + +$	0	$- - - - - -$	
$f''(x)$		$- - - - - -$	$\vert$	$- -$	$-$	$- - - - -$	
$f(x)$	$+\infty$	$\searrow$ $\cap$	0 min	$\cap$ $\nearrow$	1 max	$\searrow$ $\cap$	$-\infty$

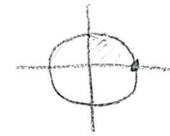
x		0		1	
$4-4x^{1/5}$	$+$	$+$	$+$	0	$- -$
$x^{1/5}$	$- -$	0	$+ +$	$+$	$+ +$
$f'(x)$	$-$	$\vert$	$+$	0	$-$

Concavity: $f''(x) = -\frac{4}{5}x^{-6/5}$

$$= -\frac{4}{5}\frac{1}{\sqrt[5]{x^6}} < 0 \text{ always.}$$



③ $f(x) = x + \cos(x)$ on $[0, 2\pi]$.



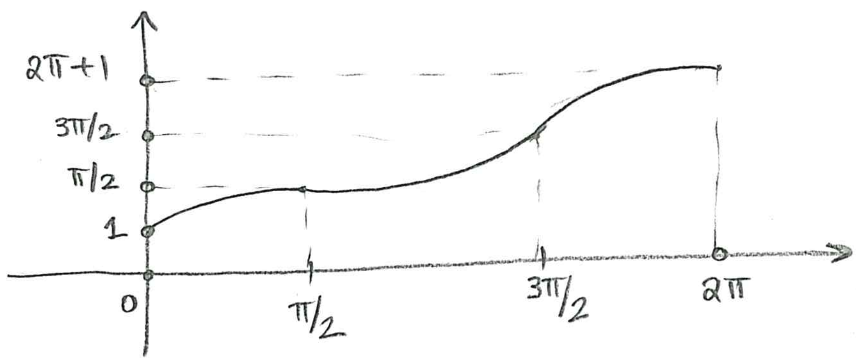
$f'(x) = 1 - \sin(x)$

$f'(x) = 0$ on $[0, 2\pi]$ whenever $\sin(x) = 1$, so at $x = \pi/2$

$f''(x) = -\cos(x)$ → changes sign at $x = \pi/2$ & $x = 3\pi/2$

x	0		$\frac{\pi}{2}$		$\frac{3\pi}{2}$		2π
$f'(x)$		$+ + + +$	0	$+ + + +$	$+$	$+ + + +$	
$f''(x)$		$- - - -$	0	$+ + +$	0	$- - - - -$	
$f(x)$	1	$\cap$ ↗	$\pi/2$	↗ $\cup$	$3\pi/2$	↗ $\cap$	$2\pi + 1$

inflection pts. at $\pi/2$ & $3\pi/2$



④ $f(x) = \dfrac{-x^3}{x^2-12}$

domain: $(-\infty, -\sqrt{12}) \cup (-\sqrt{12}, \sqrt{12}) \cup (\sqrt{12}, \infty)$

monotonicity: $f'(x) = \dfrac{-3x^2(x^2-12) + x^3(2x)}{(x^2-12)^2} = \dfrac{-3x^4 + 36x^2 + 2x^4}{(x^2-12)^2}$

$= \dfrac{-x^4 + 36x^2}{(x^2-12)^2}$ → $x^2(-x^2+36)$

$(x^2-12)^2$ → always ⊕

concavity: $f''(x) = \dfrac{(-4x^3+72x)\cdot(x^2-12)^{\cancel{2}\,1} - (-x^4+36x^2)\cdot 2\cancel{(x^2-12)}\cdot 2x}{(x^2-12)^{\cancel{4}\,3}}$

$= \dfrac{(-4x^3+72x)(x^2-12) - 4x(-x^4+36x^2)}{(x^2-12)^3}$

$= \dfrac{-24x(x^2+36)}{(x^2-12)^3}$

$-\cancel{4x^5} + 48x^3 + 72x^3 - 864x$
$+\cancel{4x^5} - 144x^3$
$= -24x^3 - 864x$
$= -24x(x^2+36)$

x		$-\sqrt{12}$		0		$\sqrt{12}$	
$-24x(x^2+36)$	+	+	+	0	−	−	−
$(x^2-12)^3$	+	0	−	−	−	0	+
f''	+	|	−	0	+	|	−

limits @ ±∞:

$\lim\limits_{x\to\infty} \dfrac{-x^3}{x^2-12} = \lim\limits_{x\to\infty} \dfrac{-x}{1-\frac{12}{x^2}} = -\infty$

$\lim\limits_{x\to-\infty} \dfrac{-x^3}{x^2-12} = \lim\limits_{x\to-\infty} \dfrac{-x}{1-\frac{12}{x^2}} = +\infty$

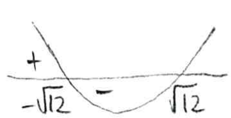
vertical asymptotes at $x = -\sqrt{12}$, $x = +\sqrt{12}$.

$\lim\limits_{x\to-\sqrt{12}^-} f(x) = \dfrac{\oplus}{0_+} = +\infty$

$\lim\limits_{x\to+\sqrt{12}^+} f(x) = \dfrac{\ominus}{0_+} = -\infty$

$\lim\limits_{x\to-\sqrt{12}^+} f(x) = \dfrac{\oplus}{0_-} = -\infty$

$\lim\limits_{x\to+\sqrt{12}^-} f(x) = \dfrac{\ominus}{0_-} = +\infty$



inflection pt.

x	$-\infty$		-6		$-\sqrt{12}$		0		$\sqrt{12}$		6		∞
$f'(x)$		$- - -$	0	$+ +$		$+ + +$	$+$	$+ +$		$+ +$	0	$- - - -$	
$f''(x)$		$+ + +$	$+$	$+ +$		$- - -$	0	$+ +$		$- -$	$-$	$- - -$	
$f(x)$	$+\infty$	$\searrow$	9	$\nearrow$	$+\infty$ | $-\infty$	$\nearrow$	0	$\nearrow$	$+\infty$ | $-\infty$	$\nearrow$	9	$\searrow$	$-\infty$

<u>Slant Asymptotes:</u>

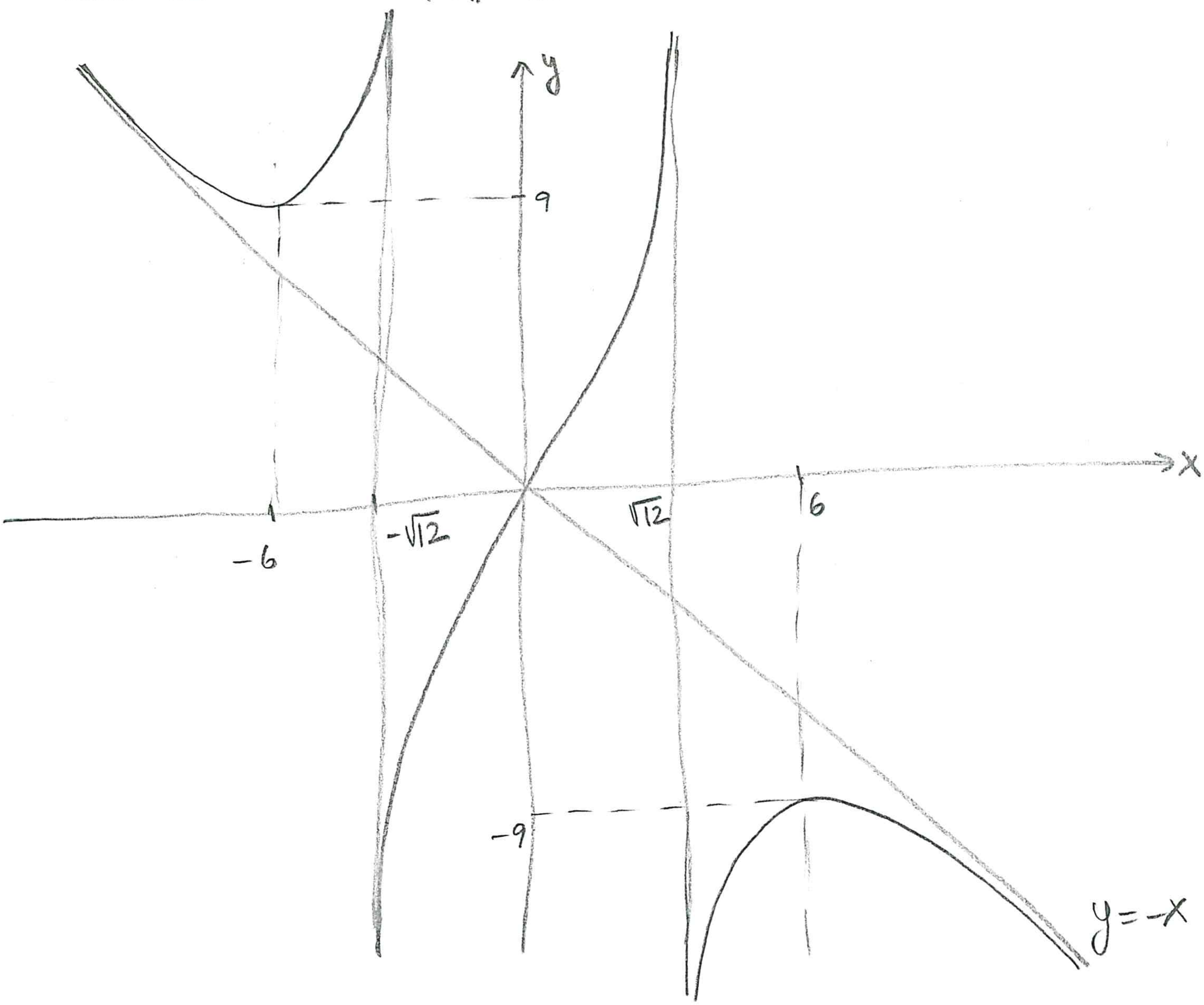
$$\begin{array}{r|l} x^3 & x^2-12 \\ \hline -x^3+12x & x \\ \hline +12x & \end{array}$$

$$f(x) = \frac{-x^3}{x^2-12} = \frac{-x(x^2-12)-12x}{x^2-12}$$

$$= -x - \frac{12x}{x^2-12} \to 0$$

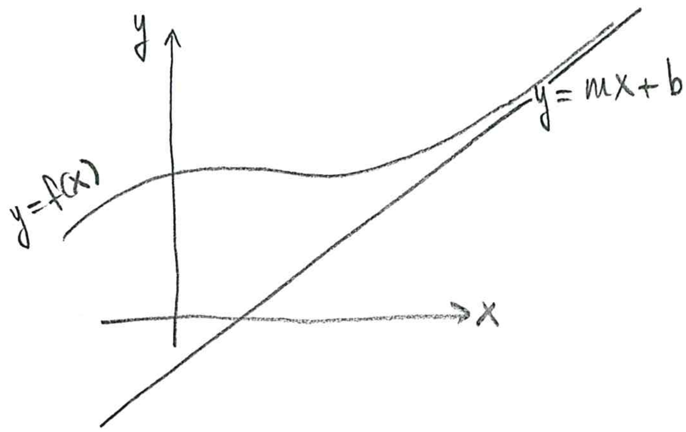
$y = -x$ slant asymptote.

<u>f is odd:</u> $f(-x) = \dfrac{-(-x)^3}{(-x)^2-12} = \dfrac{x^3}{x^2-12} = -f(x).$



⑤ $f(x) = \dfrac{8x^2}{2x-1}$

Slant asymptotes ?



$$\lim_{x \to \infty} \left[f(x) - (mx+b) \right] = 0$$

Long division:

$$\begin{array}{r|l} 8x^2 & 2x-1 \\ \hline -8x^2+4x & 4x+2 \\ \hline 4x & \\ -4x+2 & \\ \hline 2 & \end{array}$$

$$8x^2 = (2x-1)(4x+2) + 2$$

$$f(x) = \frac{8x^2}{2x-1} = \frac{(2x-1)(4x+2)+2}{2x-1} = 4x+2 + \frac{2}{2x-1}$$

$$\Rightarrow f(x) - (4x+2) = \frac{2}{2x-1} \xrightarrow[x \to \infty]{} 0$$
$x \to -\infty$

Vertical Asymptotes ? $x = \frac{1}{2}$

Horizontal Asymptotes? $\lim_{x \to \infty} \frac{8x^2}{2x-1} = \lim_{x \to -\infty} \frac{8x^2}{2x-1} = \infty$ so none